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Inequalities for Legendre Functions and Gegenbauer Functions

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In this paper three new and simple bounds for the Legendre functions of the first kind $P_\nu^{\ \mu}(x)$ for real $x \in [-1, 1]$ are proved. They can easily be transformed for an application to Gegenbauer functions. At first, a short summary of well-known inequalities of $P_\nu^{-\mu}(x)$ is given. Then a bound is derived that seems to be completely new. Finally, improvements of two known inequalities are presented.

INTRODUCTION

Upper limits on Legendre functions of the first kind $P_n^m(x)$ for real $x \in [-1, 1]$ and $n, m \in \mathbb{N}$ are essential to an investigation of the convergence and growth properties of spherical harmonic expansion, which appear very often in theoretical physics. In the following we prove simple bounds for Legendre functions of the first kind $P_\nu^{-\mu}(x)$ for real variable x as well as real parameters ν, μ, which seem to be unknown in the literature. Since the Gegenbauer functions $C_\alpha^\lambda(x)$ of real parameters α and λ, respectively, the ultraspherical polynomials $C_n^\lambda(x)$ of degree $n \in \mathbb{N}$ are closely related to the Legendre functions, see Eq. (A4), quite analogous bounds can be proved for these functions too.

A widely known inequality, which is usually cited in mathematical handbooks, e.g., [1, 2], and standard books on Legendre functions [3], is for $\nu \geqslant 1$, $\mu \geqslant 0$ with $\nu - \mu + 1 > 0$ and for $x \in [-1, 1]$ given by

$$|P_\nu^{\pm\mu}(x)| < \sqrt{\frac{8}{\pi\nu}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{(1 - x^2)^{\mu/2 + 1/4}}. \tag{1}$$

This inequality becomes, however, very crude for increasing ν and μ. For

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Legendre polynomials, i.e., for $\mu=0$ and $\nu=n\in\mathbb{N}$, there exists an improvement of Eq. (1) by the so-called Bernstein inequality, see Refs. [4, 5],

$$|P_n(x)|<\sqrt{\frac{2}{\pi n}}\frac{1}{(1-x^2)^{1/4}}. \tag{2}$$

On the other hand, for ultraspherical polynomials, it has been shown in [6] that

$$|C_n^\lambda(x)|<\frac{2^{1-\lambda}}{\Gamma(\lambda)(n+\lambda)^{1-\lambda}}\frac{1}{(1-x^2)^{\lambda/2}}, \tag{3}$$

for $n=0, 1, 2, \ldots$ and the restricted parameter interval $0<\lambda<1$. For $\lambda=\frac{1}{2}$, where, according to Eq. (A4), the ultraspherical polynomials are identical to the Legendre polynomials, Eq. (3) also provides a refinement of Bernstein's inequality (2), which has been proved previously in [7, 8].

However, all these above-mentioned inequalities have the disadvantage that they tend to infinity for $x\to 1$. A limit for the ultraspherical polynomials, that remains finite in the interval $x\in[-1, 1]$, has been presented in [9],

$$|C_n^\lambda(x)|\leqslant C_n^\lambda(1)\left[1+\frac{C_n^{\lambda'}(1)}{\lambda C_n^\lambda(1)}(1-x^2)\right]^{-\lambda/2}, \tag{4}$$

where $n=0, 1, 2, \ldots$ and $0.123\leqslant\lambda<1$. In the special case of the Legendre polynomials ($\lambda=\frac{1}{2}$) this inequality has previously been proved in [10].

In Ref. [11] a constant bound of the Legendre functions valid for all $x\in[-1, 1]$ and any integer n and m with $1\leqslant|m|\leqslant n$ is given by

$$|P_n^m(x)|\leqslant\sqrt{\frac{\Gamma(n+m+1)}{2\Gamma(n-m+1)}}. \tag{5}$$

In the following, we present in Theorem 1 two inequalities for the Legendre functions $P_\nu^{-\mu}(x)$ and the Gegenbauer functions $C_\alpha^\lambda(x)$ that are finite and valid in a wide range of the parameters ν and μ respectively α and λ. In Theorem 2 a significant improvement of (1) for $\nu, \mu\in\mathbb{N}$ is given. And finally, in Theorem 3 we prove a refinement of the bound in (5).

Results

THEOREM 1. *Let be $\nu, \mu\in\mathbb{R}$ with $\mu>-\frac{1}{2}$ and $\nu-\mu\geqslant 2$. Furthermore, let the real variable be $x\in[-1, 1]$, if $\nu-\mu\in\mathbb{Z}$; otherwise let it be $x\in[0, 1]$. Then the Legendre functions satisfy the inequalities*

$$\begin{aligned}|P_\nu^{-\mu}(x)|&\leqslant(1-x^2)^{\mu/2}\left[x^2a_\mu^{2/(\nu-\mu)}+(1-x^2)\,b_{\nu,\mu}^{2/(\nu-\mu)}\right]^{(\nu-\mu)/2}\\&\leqslant(1-x^2)^{\mu/2}\left[x^2a_\mu+(1-x^2)\,b_{\nu,\mu}\right],\end{aligned} \tag{6}$$

where

$$a_\mu := \frac{1}{2^\mu \Gamma(\mu+1)}, \qquad b_{\nu,\mu} := \frac{\Gamma(\nu-\mu+1)}{2^\nu \Gamma((\nu+\mu)/2+1)\,\Gamma((\nu-\mu)/2+1)}.$$

Proof. The integral representation (A1) of the Legendre functions leads, under the above-mentioned conditions, to the inequality

$$\begin{aligned}
|P_\nu^{-\mu}(x)| &\leqslant \frac{(1-x^2)^{\mu/2}}{\sqrt{\pi}2^\mu \Gamma(\mu+1/2)} \int_0^\pi |x + i\sqrt{1-x^2}\cos\varphi|^{\nu-\mu} \sin^{2\mu}\varphi\, d\varphi \\
&= \frac{(1-x^2)^{\mu/2}\, 2}{\sqrt{\pi}2^\mu \Gamma(\mu+1/2)} \int_0^{\pi/2} (x^2 \sin^{4\mu/(\nu-\mu)}\varphi \\
&\quad + (1-x^2)\cos^2\varphi \sin^{4\mu/(\nu-\mu)}\varphi)^{(\nu-\mu)/2}\, d\varphi \\
&\leqslant \frac{(1-x^2)^{\mu/2}}{\sqrt{\pi}2^\mu \Gamma(\mu+1/2)} [x^2 \tilde{a}_\mu^{2/(\nu-\mu)} + (1-x^2)\,\tilde{b}_{\nu,\mu}^{2/(\nu-\mu)}]^{(\nu-\mu)/2}. \qquad (7)
\end{aligned}$$

The last estimation results from an application of the Minkowski inequality (A8), valid for $\nu-\mu \geqslant 2$, where, see Ref. [1],

$$\tilde{a}_\mu := 2\int_0^{\pi/2} \sin^{2\mu}\varphi\, d\varphi = \frac{\sqrt{\pi}\Gamma(\mu+1/2)}{\Gamma(\mu+1)}, \qquad (8)$$

and

$$\tilde{b}_{\nu,\mu} := 2\int_0^{\pi/2} \cos^{\nu-\mu}\varphi \sin^{2\mu}\varphi\, d\varphi = \frac{\Gamma(\mu+\frac{1}{2})\,\Gamma((\nu-\mu)/2+\frac{1}{2})}{\Gamma((\nu+\mu)/2+1)}. \qquad (9)$$

With help of the Γ-function properties (A5) the first inequality of Theorem 1 is immediately reproduced.

The second, more simple inequality on the right-hand side of (6) is again an upper bound of the first one. For $\nu-\mu \geqslant 2$ the function $f(y) := y^{(\nu-\mu)/2}$ is convex for all $y \geqslant 0$, because

$$f''(y) = \frac{\nu-\mu}{2}\frac{\nu-\mu-2}{2}\, y^{(\nu-\mu)/2-2} \geqslant 0.$$

Hence, see Ref. [12], f satisfies the inequality

$$\begin{aligned}
&f(x^2 a_\mu^{2/(\nu-\mu)} + (1-x^2)\, b_{\nu,\mu}^{2/(\nu-\mu)}) \\
&\quad \leqslant x^2 f(a_\mu^{2/(\nu-\mu)}) + (1-x^2)\, f(b_{\nu,\mu}^{2/(\nu-\mu)}) \\
&\quad = x^2 a_\mu + (1-x^2)\, b_{\nu,\mu}. \qquad (10)
\end{aligned}$$

Since the left-hand side of (10) is identical to the bracket expression of the first inequality on the right-hand side of (6), the proof of Theorem 1 is completed.

From these results several conclusions can be drawn.

COROLLARIES. 1. *Let* $\alpha, \lambda \in \mathbb{R}$ *with* $\lambda > 0$ *and* $\alpha \geqslant 2$. *Furthermore, let* $x \in [-1, 1]$, *if* $\alpha \in \mathbb{Z}$; *otherwise let* $x \in [0, 1]$. *Then the Gegenbauer functions satisfy the inequalities*

$$|C_\alpha^\lambda(x)| \leqslant [x^2 c_{2\alpha, 2\lambda}^{2/\alpha} + (1-x^2)\, c_{\alpha,\lambda}^{2/\alpha}]^{\alpha/2} \leqslant x^2 c_{2\alpha, 2\lambda} + (1-x^2)\, c_{\alpha,\lambda}, \tag{11}$$

where

$$c_{\alpha,\lambda} := \frac{\Gamma(\alpha/2+\lambda)}{\Gamma(\lambda)\, \Gamma(\alpha/2+1)}.$$

(*This result follows immediately from Theorem* 1 *with help of Eqs.* (A4) *and* (A5).)

2. *For all* $x \in [-1, 1]$ *and all* $n, m = 0, 1, 2, \ldots$ *with* $n - m \geqslant 2$ *the integer order Legendre functions satisfy the inequalities*

$$\begin{aligned} |P_n^m(x)| &\leqslant (1-x^2)^{m/2}\, [x^2 \hat{a}_{n,m}^{2/(n-m)} + (1-x^2)\, \hat{b}_{n,m}^{2/(n-m)}]^{(n-m)/2} \\ &\leqslant (1-x^2)^{m/2}\, [x^2 \hat{a}_{n,m} + (1-x^2)\, \hat{b}_{n,m}], \end{aligned} \tag{12}$$

where

$$\begin{aligned} \hat{a}_{n,m} &:= \frac{2^{-m}\Gamma(n+m+1)}{\Gamma(m+1)\, \Gamma(n-m+1)}, \\ \hat{b}_{n,m} &:= \frac{2^{-n}\Gamma(n+m+1)}{\Gamma((n+m)/2+1)\, \Gamma((n-m)/2+1)}. \end{aligned} \tag{13}$$

(*This result follows immediately from Theorem* 1 *and Eq.* (A2).) *For the two remaining cases*: $m = n-1$ *and* $m = n$ *the integral in Eq.* (A1) *yields directly the simple results*

$$\begin{aligned} |P_n^{n-1}(x)| &= (1-x^2)^{(n-1)/2}\, |x|\, \hat{a}_{n,n-1}, \\ |P_n^n(x)| &= (1-x^2)^{n/2}\, \hat{a}_{n,n}. \end{aligned} \tag{14}$$

3. *For the Legendre polynomials* $P_n(x) \equiv P_n^0(x)$ *inequality* (12) *yields for* $n \geqslant 2$

$$|P_n(x)| \leqslant [x^2 + (1-x^2)\, \hat{b}_{n,0}^{2/n}]^{n/2} \leqslant x^2 + (1-x^2)\, \hat{b}_{n,0}, \tag{15}$$

where

$$\hat{b}_{n,0} = \frac{\Gamma(n+1)}{2^n \Gamma^2(n/2+1)} < \sqrt{\frac{2}{\pi n}}.$$

(*The estimation of* $\hat{b}_{n,0}$ *immediately results using the Stirling formula approximation of the* Γ*-function in* (A6).) *Since, unless* $x = \pm 1$, *the value in the brackets of* (15) *is smaller than* 1, *the first inequality reflects the correct asymptotic behavior of the Legendre polynomials for* $n \to \infty$.

4. *A further simplification of the inequality in Theorem* 1, *that becomes useful for variables* x *close to* ± 1, *is for all* $\nu, \mu \in \mathbb{R}$ *with* $\mu > -\frac{1}{2}$ *given by*

$$|P_\nu^{-\mu}(x)| \leqslant (1-x^2)^{\mu/2}\, a_\mu, \tag{16}$$

where again $x \in [-1, 1]$, *if* $\nu - \mu \in \mathbb{Z}$, *but otherwise* $x \in [0, 1]$ *is supposed. For integers* n, m *with* $m \geqslant 0$ *this inequality reads for all* $x \in [-1, 1]$,

$$|P_n^m(x)| \leqslant (1-x^2)^{m/2}\, \hat{a}_{n,m}. \tag{17}$$

(*These results follow immediately from* (7) *using the rough estimation*

$$|x + i\sqrt{1-x^2} \cos\varphi| \leqslant 1,$$

and Eq. (8).) *Evidently, comparing* (16) *with* (A1), *the right-hand side of* (16) *and consequently also of* (6), *is an asymptotic approximation to* $|P_\nu^{-\mu}(x)|$ *for* $x \to +1$, *which means that for* $x \to +1$ *the* "$\leqslant$" *sign in* (6) *and* (16) *can be replaced by the* "$\sim$" *sign. The equality signs in* (6) *also hold for* $x = 0$, *if* $\nu - \mu$ *is an even integer. This follows immediately from* (A1) *and* (9).

The following Theorem 2 presents an improvement of (1) for integer order Legendre functions.

THEOREM 2. *For all* $x \in (-1, 1)$ *and integers* n, m *with* $n \geqslant 1$ *and* $|m| \leqslant n$,

$$|P_n^m(x)| < d_{n,m} \frac{1}{(1-x^2)^{1/8}}, \tag{18}$$

where

$$d_{n,m} := \frac{\Gamma(1/4)}{\pi} \sqrt{\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}} \frac{1}{n^{1/4}}. \tag{19}$$

Proof. Because of (A2), the proof can be confined to $m \geqslant 0$. Setting $x = x'$ the addition theorem (A3) implies that

$$\{P_n^m(x)\}^2 \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} = \frac{1}{\pi} \int_0^\pi P_n(x^2 + (1-x^2)\cos\varphi) \cos(m\varphi)\, d\varphi$$

$$\leqslant \frac{1}{\pi} \int_0^\pi |P_n(x^2 + (1-x^2)\cos\varphi)|\, d\varphi. \tag{20}$$

Using Bernstein's inequality (2),

$$\begin{aligned}
&|P_n(x^2+(1-x^2)\cos\varphi)|\\
&\quad<\sqrt{\frac{2}{\pi n}}\frac{1}{(1-x^2)^{1/4}}\frac{1}{(\sin^2\varphi+x^2(1-\cos\varphi)^2)^{1/4}}\\
&\quad\leqslant\sqrt{\frac{2}{\pi n}}\frac{1}{(1-x^2)^{1/4}}\frac{1}{\sqrt{\sin\varphi}}.
\end{aligned}$$

Insertion of this result in (20) and evaluation of the φ-integral with help of (8) finally verifies Theorem 2.

As long as x is not too close to ± 1, (18) yields a good estimation of $P_n^m(x)$. On the other hand, inequality (17) is a good estimation near $x=\pm 1$. Hence, for $1\leqslant m\leqslant n$,

$$|P_n^m(x)|\leqslant \max_{x\in[-1,1]}\ \min\left\{\frac{d_{n,m}}{(1-x^2)^{1/8}},\ (1-x^2)^{m/2}\,\hat{a}_{n,m}\right\} \tag{21}$$

should give a proper constant bound of the integer order Legendre functions. The estimation of this maximum leads to the following theorem, which gives a refinement of (5).

THEOREM 3. *For all $x\in[-1,1]$ and all $n, m\in\mathbb{N}$ with $m\leqslant n$,*

$$|P_n^m(x)|<\frac{\Gamma(1/4)\,e^{1/4}}{\pi}\sqrt{\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}}\left(\frac{1}{2n}+\frac{1}{2m}\right)^{1/(4+1/m)}. \tag{22}$$

Proof. 1. *Case $m<n$.* Let us first estimate the relation between $\hat{a}_{n,m}$ defined in (13) and $d_{n,m}$ defined in (19). With (A7),

$$\begin{aligned}
\frac{\hat{a}_{n,m}}{d_{n,m}}&=\frac{\pi}{\Gamma(1/4)}\sqrt{\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}}\frac{n^{1/4}2^{-m}}{\Gamma(m+1)}\\
&=\frac{\pi n^{1/4}}{\Gamma(1/4)}\sqrt{\binom{n+m}{n-m}\binom{2m}{m}4^{-m}}\\
&\geqslant\frac{\pi}{\sqrt{2}\Gamma(1/4)}\sqrt{\binom{n+m}{n-m}\left(\frac{n}{m}\right)^{1/4}}\\
&>\frac{\pi}{\sqrt{2}\Gamma(1/4)}\sqrt{\binom{3}{1}}>1,
\end{aligned} \tag{23}$$

because $n+m\geqslant 3$ in this case. On the other hand, since

$$\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}=\prod_{k=1}^{2m}(n-m+k)\leqslant\prod_{k=1}^{2m}(n+m)=(n+m)^{2m},$$

we find estimating $\Gamma(m+1)$ by (A6)

$$\frac{\hat{a}_{n,m}}{d_{n,m}} < \frac{\pi}{\Gamma(1/4)} \frac{(n+m)^m e^m n^{1/4}}{2^m m^m}. \tag{24}$$

Evidently, $d_{n,m}/(1-x^2)^{1/8}$ is a strictly decreasing function of $1-x^2$ whereas $\hat{a}_{n,m}(1-x^2)^{m/2}$ is a strictly increasing one. Consequently, taking (23) into account, there exists exactly one common point of both functions in the interval $[0, 1]$ at

$$1-x_c^2 = \left(\frac{d_{n,m}}{\hat{a}_{n,m}}\right)^{8/(4m+1)},$$

the function value of which just corresponds to the expression on the right-hand side of (21). Hence, from (21) and with (19) and (24),

$$\begin{aligned} |P_n^m(x)| &\leqslant \frac{d_{n,m}}{(1-x_c^2)^{1/8}} = d_{n,m}\left(\frac{\hat{a}_{n,m}}{d_{n,m}}\right)^{1/(4m+1)} \\ &< \sqrt{\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}} \left(\frac{\Gamma(1/4)\, e^{1/4}}{\pi}\right)^{4m/(4m+1)} \left(\frac{n+m}{2nm}\right)^{m/(4m+1)} \\ &< \frac{\Gamma(1/4)\, e^{1/4}}{\pi} \sqrt{\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}} \left(\frac{n+m}{2nm}\right)^{1/(4+1/m)}, \end{aligned}$$

which proves Theorem 3 for $n > m$.

2. *Case* $m = n$. Equations (14) and (13) imply that for $n \geqslant 1$,

$$\begin{aligned} |P_n^n(x)| &\leqslant \hat{a}_{n,n} = \sqrt{\Gamma(2n+1)} \sqrt{\binom{2n}{n} 4^{-n}} < \frac{\sqrt{\Gamma(2n+1)}}{n^{1/4}\pi^{1/4}} \\ &< \frac{\sqrt{\Gamma(2n+1)}}{n^{1/(4+1/n)}} \frac{\Gamma(1/4)\, e^{1/4}}{\pi}, \end{aligned}$$

where (A7) has been applied. The last expression in this inequality chain is, however, identical to the expression on the right-hand side of (22) for $n = m$. This, finally, completes the proof of Theorem 3.

APPENDIX: Basic Formulas

In the following, a collection of basic formulas used in the main part of this work is listed.

For $\nu, \mu \in \mathbb{R}$ with $\mu > -\frac{1}{2}$, and for real $x \in [0, 1]$ the Legendre functions of the first kind have the integral representation, see Refs. [1, 3],

$$P_\nu^{-\mu}(x) = \frac{(1-x^2)^{\mu/2}}{\sqrt{\pi}2^\mu \Gamma(\mu+1/2)} \times \int_0^\pi (x + i\sqrt{1-x^2}\cos\varphi)^{\nu-\mu} \sin^{2\mu}\varphi \, d\varphi. \tag{A1}$$

The validity of Eq. (A1) can be extended to all $x \in [-1, 1]$, if $\nu - \mu \in \mathbb{Z}$.

For integer order $\mu = m = 0, \pm 1, \pm 2, \ldots$ the Legendre functions satisfy, see Refs. [1, 2],

$$P_\nu^{-m}(x) = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_\nu^m(x). \tag{A2}$$

For integer order m and integer degree $\nu = n = 0, 1, 2, \ldots$ we have $P_n^m(x) \equiv 0$, if $|m| > n$. For $m = 0$, the functions $P_n^0(x) =: P_n(x)$ are the Legendre polynomials, which satisfy the addition theorem, see Refs. [1–3],

$$P_n(xx' + \sqrt{1-x^2}\sqrt{1-x'^2}\cos\varphi) = P_n(x)\, P_n(x') + 2 \sum_{k=1}^{n} \frac{\Gamma(n-k+1)}{\Gamma(n+k+1)} P_n^k(x)\, P_n^k(x') \cos(k\varphi). \tag{A3}$$

There is a close relation between the Legendre functions $P_\nu^{-\mu}(x)$ and the Gegenbauer functions $C_\alpha^\lambda(x)$, see Refs. [1, 2],

$$C_\alpha^\lambda(x) = \frac{\Gamma(\alpha + 2\lambda)\, \Gamma(\lambda + 1/2)\, 2^{\lambda - 1/2}}{\Gamma(2\lambda)\, \Gamma(\alpha + 1)(1-x^2)^{\lambda/2 - 1/4}} P_{\alpha+\lambda-1/2}^{-(\lambda-1/2)}(x). \tag{A4}$$

Consequently, using Eq. (A4), many results derived for the Legendre functions can easily be adapted to the Gegenbauer functions.

The recurrence and duplication formulas of the Γ-function read, see Refs. [1, 2],

$$\begin{aligned} \Gamma(x+1) &= x\Gamma(x) \\ \Gamma(2x) &= \pi^{-1/2} 2^{2x-1} \Gamma(x)\, \Gamma(x + 1/2). \end{aligned} \tag{A5}$$

An upper and lower bound of the Γ-function is given for $x \geqslant 1$ by the Stirling formula in the form, see Ref. [12],

$$\exp\left(\frac{1}{12x+1}\right) < \frac{\Gamma(x+1)}{\sqrt{2\pi x^x e^{-x}}\sqrt{x}} < \exp\left(\frac{1}{12x}\right). \tag{A6}$$

For all $n \in \mathbb{N}$, the binomial coefficients satisfy

$$\frac{4^n}{\sqrt{4n}} \leqslant \binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}}. \tag{A7}$$

The first inequality in (A7) can be found in [12]. The second one can immediately be derived using (A6).

For $p \geqslant 1$ the integral version of the Minkowski inequality reads, see Ref. [12],

$$\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \leqslant \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}, \tag{A8}$$

supposed that the integrals exist.

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